Encoding Geometry on the Geometric-Quantization Ladder: a placement, a dictionary to canonical quantum gravity, and its failures

Sebastian Zając[1]

[1] *Military University of Technology (WAT), Warsaw*

(Dated: August 15, 2026)

Quantum machine learning has produced a sharp negative result about *encodings*: a family of states built from real amplitudes has an identically vanishing Berry connection, its pulled-back Fubini–Study metric is (one quarter of) the Fisher–Rao metric of classical statistics, and its image is a Lagrangian locus carrying no symplectic structure. This note asks what survives when the same bookkeeping is applied to canonical quantum gravity, where the Wheeler–DeWitt operator is likewise real. It is a dictionary note, and its usefulness lies as much in its negative entries as in its positive ones. We do not claim that the reality of the constraint explains the frozen formalism: the constraint freezes the state whether or not it is real, and a complex WKB solution is exactly as stationary as a real one. What transfers as theorems is modest and stated precisely: the Fisher–Rao collapse for real families; a one-line lemma that the velocity generated by a real operator on a real state is orthogonal to the real locus, so the would-be direction of evolution is invisible to any pullback; and the identification of the Hilbert-space problem with a choice of complex structure. What fails is stated with equal care: there is no canonical family of states in quantum gravity to play the role of a dataset, no probability measure on superspace, a normal-bundle mismatch, and no CCR algebra to support representation-theoretic arguments. The organizing statement we propose is a *placement*: the two literatures occupy two different levels of one geometric-quantization scheme. The encoding programme lives at the prequantum level, where the relevant datum is the curvature of the Berry connection and the classical/quantum distinction is $F \neq 0$ versus $F = 0$. Semiclassical gravitational time lives one level down, on a Lagrangian leaf, where the restricted connection is necessarily flat and its entire content is holonomy — which is precisely the Bohr–Sommerfeld setting, with the Maslov index as its standard caustic correction. We emphasize that this flatness is not a discovery of ours but textbook geometric quantization, and that Berry connections in quantum cosmology have been used before; the contribution claimed here is only the placement of the encoding programme's negative result on the same ladder, together with an explicit ledger of what does and does not transfer.

I. WHAT KIND OF NOTE THIS IS

This is a dictionary note between two literatures: the geometry of quantum encodings, developed for machine learning [26, 27], and canonical quantum gravity. Its purpose is to establish where the correspondence is a theorem, where it is an analogy, and where it fails; and then to locate both literatures within a single, older framework — geometric quantization — in which their relationship becomes a matter of *level* rather than similarity.

Three caveats belong at the front, because each corrects a claim made in an earlier draft of this note.

(i) Reality does not explain timelessness. One might hope that the reality of the Wheeler–DeWitt operator accounts for the absence of time in

$$\hat{\mathcal{H}}\,\Psi[g_{ij},\phi] \;=\; 0. \tag{1}$$

It does not. Equation (1) is a constraint and freezes the state regardless of whether Ψ is real: a complex WKB solution $e^{iS_0/\hbar}\chi$ annihilated by $\hat{\mathcal{H}}$ is exactly as stationary as a real one. Timelessness is caused by the constraint. What reality does is distinct, and worth isolating: it removes the geometric structures one would use to *describe* a flow even if one had it. [E]

(ii) The flatness of the WKB connection is not new. Section V observes that the connection attached to the WKB phase is exact, hence flat, so that its content is holonomy rather than curvature. This is the standard structure of Bohr–Sommerfeld theory in geometric quantization [14–16]: on a Lagrangian submanifold the symplectic form vanishes, any symplectic potential restricts to a closed form, the restricted prequantum connection is flat, and a global covariantly constant section exists iff the holonomy is trivial. The Maslov index [17, 18] is the classical caustic correction within that same framework. We claim no novelty for this and cite it as background. [K]

(iii) Berry connections in quantum cosmology are not new either. Rotondo and Nambu [19] analyse an FLRW minisuperspace model in which the evolution of the geometry with respect to a matter clock is governed by a (matrix-valued) Berry connection on the reduced geometrodynamical space, with the standard Schrödinger equation recovered

when that connection vanishes. Their construction is closer to the present circle of ideas than anything else we have found, and the reader should regard the present note as complementary to it rather than independent of it. [K]

What remains, and what we do claim, is the *placement* of Sec. V together with the ledger of Secs. III and IV. Claims are tagged [E] established here or standard and proved, [K] known in the cited literature and used as background, [P] plausible with an identified proof route, [C] conjectural.

II. THE ENCODING SIDE

Let $\mathcal{H}$ be a complex Hilbert space, $P(\mathcal{H})$ its projectivization with the Fubini–Study Kähler structure $h_{\mathrm{FS}} = g_{\mathrm{FS}} + i\,\omega_{\mathrm{FS}}$ [1], and let $\varphi : \Sigma \to P(\mathcal{H})$, $\lambda \mapsto [\psi_\lambda]$, be a smooth normalized family (an *encoding*). Local geometry is carried by the quantum geometric tensor [2]

$$\begin{gathered} Q_{\mu\nu} = \langle \partial_\mu \psi | \big(\mathbb{1} - |\psi\rangle\langle\psi|\big) | \partial_\nu \psi \rangle, \\ \varphi^* g_{\mathrm{FS}} = \operatorname{Re} Q, \qquad \varphi^* \omega_{\mathrm{FS}} = dA, \qquad A_\mu = i\langle \psi | \partial_\mu \psi \rangle. \end{gathered} \tag{2}$$

Since $\dim \Sigma$ is typically far below $\dim_{\mathbb{R}} P(\mathcal{H})$, φ is an immersion: φ_* injective, φ^* surjective with kernel, the mismatch being the normal bundle. Pullback is intrinsic and reports nothing about normal directions. [E]

Fix an antilinear involution C with real subspace $\mathcal{H}_{\mathbb{R}}$; the real locus in $P(\mathcal{H})$ is $\mathbb{RP}^{N-1}$.

Lemma 1 (Collapse on the real locus)**.** *For a smooth normalized family with $\psi_\lambda \in \mathcal{H}_{\mathbb{R}}$: [E] (i) $A_\mu = 0$ identically; (ii) $Q_{\mu\nu}$ is real symmetric, hence $\varphi^*\omega_{\mathrm{FS}} = 0$ and the real locus is Lagrangian; (iii) writing $\psi_i = \sqrt{p_i}$, $\varphi^* g_{\mathrm{FS}} = \frac{1}{4} g_{\mathrm{FR}}$, the Fisher–Rao metric — the real-amplitude case of the general relation between the Fubini–Study/Bures metric and quantum Fisher information [3, 4].*

Proof. (i) Differentiating $\langle\psi|\psi\rangle = 1$ gives $\langle \partial_\mu \psi|\psi\rangle + \langle \psi|\partial_\mu\psi\rangle = 0$; for a real family the two terms are equal and real, so each vanishes. (ii) With $\psi, \partial_\mu\psi$ real every term in Eq. (2) is real, so $\operatorname{Im} Q = 0$, hence $\varphi^*\omega_{\mathrm{FS}} = 0$; and since $\dim \mathbb{RP}^{N-1} = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{CP}^{N-1}$, a half-dimensional isotropic locus is Lagrangian. (iii) $\partial_\mu \sqrt{p_i} = \partial_\mu p_i / (2\sqrt{p_i})$ gives $\operatorname{Re} Q_{\mu\nu} = \frac{1}{4}\sum_i \partial_\mu p_i \partial_\nu p_i / p_i$, the subtracted term vanishing by (i). $\square$

The machine-learning consequence is that real encodings are classical statistical models: their kernel is a classical inner-product kernel, and no data-independent unitary changes this [26].

III. WHAT TRANSFERS

A. Reality of the constraint, and the collapse

In the metric representation,

$$\hat{\mathcal{H}} = -\hbar^2 G_{ijkl} \frac{\delta^2}{\delta g_{ij} \delta g_{kl}} - \sqrt{g}\big({}^{(3)}R - 2\Lambda\big) + \hat{\mathcal{H}}_{\mathrm{matter}} \tag{3}$$

is of Klein–Gordon type on superspace: second order, no first-order $i\partial$ term. It commutes with the complex conjugation C supplied by that representation, admits a complete set of real solutions, and the leading candidate states are real — Hartle–Hawking [6] by construction, Vilenkin-type [7] up to the decaying/growing decomposition. [E]

C is not an arbitrary choice among the infinitely many antilinear involutions on $\mathcal{H}$: it is the one distinguished by the representation in which Eq. (3) is written, and $\hat{\mathcal{H}}$ is real precisely with respect to it. All statements below are relative to that C; nothing is claimed to be representation-independent. [E]

Lemma 1 then applies verbatim to any smooth family of real solutions. Whether this deserves the name "the universe is a classical statistical model" is taken up in Failures 1 and 2.

B. The would-be flow is orthogonal to the real locus

Lemma 2 (Normality)**.** *Let H be self-adjoint with $CHC = H$, and $\psi \in \mathcal{H}_{\mathbb{R}}$ normalized. Then $v = -\frac{i}{\hbar} H\psi$ satisfies $Cv = -v$, so $\operatorname{Re}\langle u, v\rangle = 0$ for every $u \in \mathcal{H}_{\mathbb{R}}$; in particular $v \perp T_\psi \varphi(\Sigma)$ for every real family φ. [E]*

Proof. $H\psi \in \mathcal{H}_\mathbb{R}$ since H and ψ are real; multiplication by $-i/\hbar$ maps $\mathcal{H}_\mathbb{R}$ into $i\mathcal{H}_\mathbb{R}$, and $\mathrm{Re}\langle u, iw\rangle = 0$ for $u, w \in \mathcal{H}_\mathbb{R}$. Any real family has $T_\psi\varphi(\Sigma) \subset \mathcal{H}_\mathbb{R}$. □

Corollary 1. *For real families the flow direction generated by a real operator lies in the orthogonal complement of the image, hence is annihilated by φ^*: it is extrinsic data of the embedding, not of the induced geometry. [E]*

Two qualifications. First, Corollary 1 is logically an instance of the tautology that pullbacks do not see normal data; its content is the identification of *which* normal direction is at issue, namely $J(\mathcal{H}_\mathbb{R})$. Second, in the WDW case it is nearly vacuous on its own, because the constraint sets $v = 0$ outright; its interest is comparative, explaining why in the Schrödinger setting real states cannot evolve nontrivially. We have not verified whether Lemma 2 appears in the geometric formulation of quantum mechanics [13], where the Kähler machinery is developed in full; it is elementary enough that it may well do. [E] as mathematics, priority unverified.

Remark 1 (Two independent degeneracies). (a) *Constraint*: $\hat{\mathcal{H}}\Psi = 0$ makes every solution stationary, real or complex; this causes timelessness. (b) *Reality*: for a real family the structures with which one would describe a flow vanish identically and the flow direction is orthogonal to the family. Neither implies the other: the WKB family of Sec. V solves Eq. (1) with $A \neq 0$. [E]

C. The Hilbert-space problem is a choice of J

The Klein–Gordon pairing on superspace is indefinite and real solutions have zero norm — consistent with Lemma 1(ii), a Lagrangian locus being null for the symplectic pairing. Making it positive requires a frequency splitting, and [E]

$$\begin{aligned} &\text{positive-definite } \langle\cdot,\cdot\rangle \iff \text{frequency splitting} \\ &\iff \text{compatible } J \iff \text{timelike direction.} \end{aligned} \tag{4}$$

Theorem 1 (J and time, minisuperspace)**.** *Let (V, Ω) be the finite-dimensional real symplectic solution space of a minisuperspace WDW equation. Compatible positive complex structures J, positive-definite Hermitian inner products with imaginary part Ω, frequency splittings, and one-parameter unitary groups with a positive generator are in mutual bijection; real solutions carry none of them. [E]*

Proof. Given compatible positive J, set $\langle u, v\rangle_J = \frac{1}{2}\Omega(u, Jv) + \frac{i}{2}\Omega(u, v)$ (Ashtekar–Magnon/Kay [11, 12]); this is positive-definite Hermitian with imaginary part Ω, and the $+i$ eigenspace of J is the positive-frequency subspace. Conversely a frequency splitting determines J, and the polar decomposition of the positive generator of a one-parameter unitary group recovers J. Real solutions form the fixed locus of C, and $CJ = -JC$ implies they lie in no frequency eigenspace. □

This is the inner-product thesis of [27] transplanted: not the Born rule but the choice of Hermitian structure selects a time axis. That the choice is physically consequential is established independently by Gielen and Menéndez-Pidal [20–22], who show for FRW models with a scalar field and a perfect fluid that quantizations unitary in different clocks disagree about singularity resolution. [K]

Remark 2 (No protection from Stone–von Neumann). It is tempting to argue that in minisuperspace all admissible J are unitarily equivalent by Stone–von Neumann, hence the choice is harmless. The argument is wrong and we record the error because an earlier draft made it. Stone–von Neumann concerns representations of the CCR algebra for a *fixed* symplectic structure and says nothing about the physical content of a deparametrization: different clocks give different physical Hamiltonians and different predictions, as [20] demonstrates *within* minisuperspace. [E]

IV. WHERE THE DICTIONARY FAILS

Failure of the dictionary 1 (No canonical family). In the encoding arena the family $\varphi : \Sigma \to P(\mathcal{H})$ is given by the problem: Σ is a dataset, φ a chosen feature map. In quantum gravity the solutions of Eq. (1) form a linear space and no family is distinguished; one may draw any curve through it. The Fisher–Rao geometry produced by Lemma 1 is therefore a property of the curve one elects to draw, not of the theory. Statements of the form "the wave function of the universe is a $\sqrt{p}$ encoding" require a prior specification of what plays the role of the dataset, and different choices give different geometry. We know no principled selection. [E]

Failure of the dictionary 2 (No probability measure on superspace). Lemma 1(iii) delivers the Fisher–Rao metric *of a probability family.* On full superspace $|\Psi|^2$ is not normalizable, not conserved, and has no accepted probability interpretation — this being precisely the Hilbert-space problem of Sec. III C. The phrase "classical statistical model on superspace" is thus formal; it acquires content only after a minisuperspace truncation with a self-adjoint realization of $\hat{\mathcal{H}}$ on an auxiliary L^2 space [8], i.e. exactly where the motivating problem is least severe. [E]

Failure of the dictionary 3 (Normal bundle mismatch). For the real locus, $N(\mathbb{RP}^{N-1}) = J(T\mathbb{RP}^{N-1})$, a Lagrangian statement. For a low-dimensional image $\varphi(\Sigma)$ the normal space has dimension $2N-2-\dim\Sigma$, of which $J(T\varphi(\Sigma))$ is a small subspace. Lemma 2 survives — it asserts only orthogonality of v to $\mathcal{H}_{\mathbb{R}} \supset T\varphi(\Sigma)$ — but any argument identifying "the normal bundle" with "the phase directions" holds for the real locus and fails for a generic image. [E]

Failure of the dictionary 4 (No CCR algebra on superspace). Shale/Haag-type arguments presuppose a linear symplectic space with a well-defined CCR algebra and Fock construction. Superspace supplies none unambiguously. Claims that the multiple-choice problem "is a Haag phenomenon" are structural analogies, not theorems, and the stronger formulation of an earlier draft is withdrawn. [P] at best.

V. PLACEMENT: TWO LEVELS OF ONE GEOMETRIC-QUANTIZATION SCHEME

A. The observation

In the Born–Oppenheimer/WKB expansion $\Psi = e^{iS_0[g]/\hbar}\chi[g;\phi] + \dots$ [9, 10] the gravitational eikonal satisfies the Hamilton–Jacobi form of the Einstein equations,

$$\tfrac{1}{2}G_{ijkl}\frac{\delta S_0}{\delta g_{ij}}\frac{\delta S_0}{\delta g_{kl}} - \sqrt{g}\big({}^{(3)}R - 2\Lambda\big) = 0, \tag{5}$$

and matter obeys a Schrödinger equation along $\partial_\tau = G\,\nabla S_0 \cdot \nabla$. [E]

Observation 1 (Flatness, and its standard name)**.** *The connection attached to the WKB phase is $A = dS_0$ up to normalization, which is exact; hence $F = dA = 0$ wherever S_0 is single-valued. Semiclassical time is generated by a* flat *connection, and the entire content of a flat connection is its holonomy,*

$$\mathrm{hol}:\ \pi_1(\Sigma) \to U(1), \qquad \gamma \mapsto \exp\Big(\tfrac{i}{\hbar}\oint_\gamma dS_0\Big), \tag{6}$$

an H^1 datum. This is the Bohr–Sommerfeld structure of geometric quantization*: on a Lagrangian submanifold the symplectic form vanishes, any symplectic potential restricts to a closed form, the restricted prequantum connection is flat, and a global covariantly constant section exists iff the holonomy is trivial [14–16]. The Maslov index [17, 18] is the associated caustic correction. [K]*

We stress the tag. Observation 1 contains nothing new; it is recalled because of what it implies for the dictionary.

B. Why this settles the relation between the two arenas

In the encoding arena the quantity distinguishing a quantum from a classical model is the *curvature* $\varphi^*\omega = dA$: an encoding with A exact — in particular any real encoding, where $A = 0$ — is classical, and the advantage lives in $F \neq 0$ [26]. That is a *prequantum* statement: it concerns the curvature of the prequantum connection on the state manifold.

Observation 1 says the gravitational mechanism is not of that type: it produces time from a connection with $F = 0$. The two arenas therefore do not sit at the same place, and the naive reading of the dictionary — "phase content is curvature, so gravitational time should be curvature" — is wrong. They sit at two successive levels of one scheme [P]:

The placement has three payoffs, and we state them as the note's actual claims.

(1) It corrects a level error. Chern-class (H^2) conjectures about the classicalization of the universe, of the kind the encoding programme naturally suggests, are aimed one level too high for semiclassical time. The relevant obstruction for the existence of a global time function is the holonomy (6), an H^1 datum. [E]

(2) It delimits the winding language. One is tempted to write the obstruction as $[(d\theta + A)/2\pi] \in H^1(\Sigma;\mathbb{Z})$ for a general phase; this is ill-defined unless $d\theta + A$ is closed, i.e. unless $F = 0$. Observation 1 identifies the WKB/Bohr–Sommerfeld regime as exactly where the expression is legitimate — and by the same token shows that "winding" is not available as a general diagnostic for encodings, where $F \neq 0$ is the interesting case. [E]

Level	Datum	Arena
Prequantum	curvature $F = dA$ of the Berry/prequantum connection; $F = 0$ versus $F \neq 0$	encoding geometry: $\sqrt{p}$ bottleneck, classical versus quantum feature maps [26]
Polarization / Lagrangian leaf	restricted connection is flat; the datum is holonomy in H^1, with Maslov corrections	semiclassical gravitational time: WKB phase, Bohr–Sommerfeld, caustics [14, 17]
Inner product / J	choice of compatible complex structure	the Hilbert-space problem; clock dependence [20]; Theorem 1

(3) It suggests the right diagnostic on each side. On the encoding side, curvature-based quantities (Berry curvature, and the entanglement-based proxies used in spectral data analysis) are appropriate. On the gravitational side they are not, and one should compute holonomies instead. This is a concrete methodological consequence, and Sec. VI lists the first computation it recommends.

C. Relation to existing uses of Berry connections in cosmology

Rotondo and Nambu [19] study an FLRW minisuperspace model in which the solution to Eq. (1) is conditioned on a matter clock state and the resulting evolution of the geometry is governed by a matrix-valued Berry connection on the reduced geometrodynamical space, the ordinary Schrödinger equation being recovered when the off-diagonal elements vanish (which they argue happens under decoherence in the semiclassical limit, where clock time agrees with cosmic time). Two remarks. First, this is a genuine precedent for the use of connection language in this problem and should be read alongside the present note. Second, their connection is non-abelian, hence generically *not* flat, and lives at the clock-conditioning level rather than on a Lagrangian leaf; comparing the two constructions — when does the Rotondo–Nambu connection reduce to the flat WKB one? — is a well-posed question that we have not answered. [P]

VI. WHAT IS COMPUTABLE

Only finite-dimensional items are listed, since Failures 2 and 4 block the rest.

1. *Holonomy in a recollapsing FRW model.* Compute (6) around a loop encircling the classical turning point and compare the caustic contribution with the Maslov prediction. This is the concrete version of Sec. V and yields a number; it is also the item whose absence would leave this note without any quantitative content. [P]

2. *Classify J for FRW with a massless scalar.* Enumerate compatible positive complex structures, exhibit the induced time functions, and compare with the clock choices of [20, 21]; this will decide whether the identification [clock $\leftrightarrow J$] is exact or merely suggestive. [P]

3. *Reduction of the Rotondo–Nambu connection.* Determine the conditions under which the matrix-valued clock connection of [19] degenerates to the flat WKB connection of Observation 1. [P]

4. *Page–Wootters diagnostics with finite-dimensional clocks.* For $\dim \mathcal{H}_C < \infty$ the mutual information and logarithmic negativity across the clock cut are well-defined; for an ideal clock with $\mathcal{H}_C = L^2(\mathbb{R})$ they are not, so the restriction is not cosmetic [23–25]. [P]

VII. STATUS LEDGER

Proved here, elementary [E]: Lemma 1; Lemma 2 and Corollary 1 (priority unverified against [13]); Theorem 1; Remarks 1 and 2; the four failures of Sec. IV; payoffs (1) and (2) of Sec. V B.

Known background, used not claimed [K]: flatness of the restricted connection on a Lagrangian leaf and the Bohr–Sommerfeld holonomy criterion [14–16]; the Maslov index [17, 18]; Berry connections in conditioned quantum cosmology [19]; clock dependence of quantum cosmology [20, 21]; the WKB emergence of time [9, 10].

Plausible with proof routes [P]: the three-level placement as a precise statement rather than a table; the reduction question for the Rotondo–Nambu connection; the identification of clock choices with the J-classification; all items of Sec. VI.

Withdrawn from earlier drafts: that reality of the constraint explains timelessness; that the multiple-choice problem is established to be a Haag phenomenon; that the Maslov index is a "shadow" of a winding class; that Chern-class arguments are the right level for semiclassical time; and any suggestion that Observation 1 is a new result.

Assessment

The honest summary is that the transfer from encoding geometry to canonical quantum gravity yields three elementary theorems, four instructive failures, and one organizing statement — that the two literatures occupy the prequantum and Bohr–Sommerfeld levels of a single geometric-quantization scheme — whose mathematical ingredients are all classical and whose only novelty is the identification itself. Whether that is worth a note is for the reader to judge; we have tried to make the judgement possible by labelling every claim. What the placement does buy is methodological and concrete: curvature-based diagnostics belong on the encoding side, holonomy computations on the gravitational side, and the first such computation is listed above.

---

[1] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, 2nd ed., Cambridge University Press (2017).

[2] J. P. Provost and G. Vallee, "Riemannian structure on manifolds of quantum states," Commun. Math. Phys. **76**, 289 (1980).

[3] W. K. Wootters, "Statistical distance and Hilbert space," Phys. Rev. D **23**, 357 (1981).

[4] S. L. Braunstein and C. M. Caves, "Statistical distance and the geometry of quantum states," Phys. Rev. Lett. **72**, 3439 (1994).

[5] B. S. DeWitt, "Quantum theory of gravity. I. The canonical theory," Phys. Rev. **160**, 1113 (1967).

[6] J. B. Hartle and S. W. Hawking, "Wave function of the Universe," Phys. Rev. D **28**, 2960 (1983).

[7] A. Vilenkin, "Boundary conditions in quantum cosmology," Phys. Rev. D **33**, 3560 (1986).

[8] J. J. Halliwell, "Introductory lectures on quantum cosmology," in *Quantum Cosmology and Baby Universes*, World Scientific (1991); arXiv:0909.2566.

[9] T. Banks, "TCP, quantum gravity, the cosmological constant and all that," Nucl. Phys. B **249**, 332 (1985).

[10] C. Kiefer, *Quantum Gravity*, 3rd ed., Oxford University Press (2012).

[11] A. Ashtekar and A. Magnon, "Quantum fields in curved space-times," Proc. R. Soc. Lond. A **346**, 375 (1975).

[12] B. S. Kay, "Linear spin-zero quantum fields in external gravitational and scalar fields," Commun. Math. Phys. **62**, 55 (1978).

[13] A. Ashtekar and T. A. Schilling, "Geometrical formulation of quantum mechanics," in *On Einstein's Path*, Springer (1999); arXiv:gr-qc/9706069.

[14] J. Śniatycki, *Geometric Quantization and Quantum Mechanics*, Springer (1980).

[15] N. M. J. Woodhouse, *Geometric Quantization*, 2nd ed., Oxford University Press (1992).

[16] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, American Mathematical Society (1977).

[17] V. I. Arnold, "On a characteristic class entering into conditions of quantization," Funct. Anal. Appl. **1**, 1 (1967).

[18] V. P. Maslov and M. V. Fedoriuk, *Semi-Classical Approximation in Quantum Mechanics*, Reidel (1981).

[19] M. Rotondo and Y. Nambu, "Clock time in quantum cosmology," Universe **5**, 66 (2019).

[20] S. Gielen and L. Menéndez-Pidal, "Singularity resolution depends on the clock," Class. Quantum Grav. **37**, 205018 (2020), arXiv:2005.05357.

[21] S. Gielen and L. Menéndez-Pidal, "Unitarity, clock dependence and quantum recollapse in quantum cosmology," Class. Quantum Grav. **39**, 075011 (2022), arXiv:2109.02660.

[22] L. Menéndez-Pidal, *The Problem of Time in Quantum Cosmology*, PhD thesis, University of Nottingham (2022), arXiv:2211.09173.

[23] D. N. Page and W. K. Wootters, "Evolution without evolution: Dynamics described by stationary observables," Phys. Rev. D **27**, 2885 (1983).

[24] A. R. H. Smith and M. Ahmadi, "Quantizing time: Interacting clocks and systems," Quantum **3**, 160 (2019), arXiv:1712.00081.

[25] P. A. Höhn, A. R. H. Smith, and M. P. E. Lock, "The Trinity of Relational Quantum Dynamics," Phys. Rev. D **104**, 066001 (2021), arXiv:1912.00033.

[26] S. Zając, J. L. Cybulski, B. Dziewit, and T. Kulpa, "The inverse Born rule fallacy: On the informational limits of phase-locked amplitude encoding," arXiv:2602.21350 [quant-ph].

[27] S. Zając, "The Hermitian inner product selects the time axis, the Born rule measures it," arXiv:2607.05447 [quant-ph].