Classes of Maps as the Primary Object

A Unified View of Geometry, Quantization, and Quantum Information Field Theory

Sebastian Zając
SGH Warsaw School of Economics

May 6, 2026

Abstract

We argue that the standard physics ontology of points-on-a-manifold, rigorously decomposed, is operationally equivalent to a classification of admissible classes of maps from observation data into suitable algebras of observables. Geometry, topology, classical and quantum mechanics, and Quantum Information Field Theory (QIFT) are then not different theories about different worlds, but different choices within a single morphism-primary framework. We trace the lineage of this view (Yoneda, Cartan, Gelfand, Connes, Lawvere) and list the conceptual transitions explicitly: *points* $\to$ *morphisms*, *manifold* $\to$ *frame bundle*, *space* $\to$ *algebra*, *commutative* $\to$ *noncommutative*, *particle* $\leftrightarrow$ *wave*, *Stone–von Neumann* $\to$ *Haag*, *data* $\to$ *Fock encoding*. Each transition is a refinement of the admissible class of maps. We close by positioning QIFT, and in particular the Equivalence Theorem of [12], as the operational realization of this program in quantum machine learning.

1 Motivation: the silent ontology

Standard expositions of physics open with a manifold M whose points are taken as primary, and only afterwards equip M with metric, connection, and dynamical fields. Two contemporary practices undercut this opening move:

- In general relativity, the hole argument and dynamical diffeomorphism invariance show that bare points carry no operational content; what is operationally given is the network of fields and their relations [8, 9].
- In data analysis, classical or quantum, the practitioner does not possess a manifold. They possess measurements: arrays of numbers indexed by features. Whatever "geometry" the data carries is a construct produced by a choice of representation map.

The shared structural content is the following: what is invariantly given is not the point-set, but *the class of admissible maps from data into a target algebra of observables.* Choosing a class is choosing a geometry.

2 The lineage

The morphism-primary view is not new. It is dispersed across mathematical traditions whose operational fusion remains incomplete in standard physics curricula.

- **Yoneda lemma [1].** An object X in a category $\mathcal{C}$ is determined up to isomorphism by the functor $\hom_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. *What X is = what one can do with X.*
- **Cartan's moving frames (repères mobiles).** Riemannian and pseudo-Riemannian geometry are reduction problems on the principal frame bundle. A geometry is a G-structure: a

chosen subclass of admissible frame transformations [2].

- **Sheaf and locale theory.** The smooth structure of a manifold is encoded in the sheaf $\mathcal{O}_M$ of smooth functions; the topology, in the lattice of opens, with points reconstructed as a derived (and sometimes absent) notion [3].
- **Gelfand–Naimark duality.** A commutative unital C*-algebra is the algebra of continuous functions on its spectrum. Topology and algebra are formally equivalent; the direction of priority is a choice [4].
- **Noncommutative geometry (Connes).** Drop commutativity. Classical geometry is the commutative slice of a strictly larger framework [4].
- **Topos theory and synthetic differential geometry (Lawvere).** Geometry, logic, topology, and infinitesimals are internal data of a topos. Different toposes = different "worlds" with different admissible function classes [5, 6].
- **Higher category theory and HoTT.** Identifications are first-class; numbers, types, and spaces are positions in higher-categorical structures [7].

3 The transitions, explicitly

Reading each move below as a controlled change of the admissible class of maps yields a single conceptual ladder.

1. *Points → morphisms.* Objects are their hom-functors. Numbers are positions in a structure (Dedekind); elements are arrows from the terminal object. The set-theoretic notion of "element" is recovered as a special case.

2. *Manifold → frame bundle.* Geometry is the reduction of the structure group of the frame bundle. The Lorentz group is not a coincidence arising from the linear part of a Taylor expansion; it is a G-structure datum.

3. *Space → algebra.* The Gelfand–Naimark correspondence $M \leftrightarrow C(M)$ permits algebra-first or space-first formulations indifferently. The choice is conventional in the commutative case.

4. *Commutative → noncommutative.* Releasing commutativity yields quantum geometry. Classical mechanics is the commutative slice of a strictly larger category of observable algebras.

5. *Particle ↔ wave.* Both descriptions live within the commutative class. Fourier transform is a unitary equivalence *inside* this class. Nothing in the duality is intrinsically quantum.

6. *Classical → quantum.* Quantization is a deformation $C^\infty(M) \rightsquigarrow A_\hbar$ to a noncommutative algebra. "Quantumness" resides in the choice of map class, not in any specific wavefunction.

7. *Stone–von Neumann → Haag.* For finitely many degrees of freedom, the irreducible representation of the Heisenberg algebra is unique up to unitary equivalence [10]. For infinitely many, a continuous family of inequivalent representations appears [11]. The classification of admissible classes becomes the central content.

8. *Data → Fock encoding.* Raw measurements acquire geometric content only after a choice of map into a Hilbert or Fock space. Different encoding maps construct different effective geometries on the same numerical input.

4 Particle, wave, and quantum as classes of maps

The popular dualist narrative places "particle" and "wave" as two faces of a single quantum substance. From the algebraic standpoint, both are classical idealizations: each corresponds to a commutative subalgebra of observables. Fourier transform is a unitary equivalence *within* the class of commutative algebras with translation symmetry. The genuinely quantum content is precisely what such intra-commutative equivalences cannot reach: the noncommutativity $[\hat{x}, \hat{p}] = i\hbar$, which forces a choice of map class whose structure is not preserved by any commutative reparametrization.

The Stone–von Neumann theorem then says: for finitely many degrees of freedom, the choice among admissible noncommutative classes is, up to unitary equivalence, no choice at all. Haag's theorem says: in QFT this collapses, and the classification of inequivalent classes becomes nontrivial. The shift from non-relativistic QM to QFT is, in this language, the shift from a one-class regime to a multi-class regime.

5 Application: QIFT

In Quantum Information Field Theory [12, 13] the above becomes operational rather than metaphysical.

- *Encoding = field choice.* A QML encoding is a functor from the data category to (a subcategory of) the Hilbert / Fock category. Different encodings are different functors, hence in general inequivalent classes of maps.
- *Equivalence Theorem.* Probability loading followed by any fixed unitary realizes only bilinear forms in $\sqrt{p(x)}$, collapsing the hypothesis class to one equivalent to a classical kernel method. In the present language: the entire family of such encodings lies in a single, effectively commutative class of maps [12].
- $\hbar_{\mathrm{ZC}}$ *as order parameter.* The linear-entropy quantity $\hbar_{\mathrm{ZC}} = 1 - \mathrm{Tr}\,\rho^2$ mediates the transition from a Stone–von Neumann regime (uniqueness of map class up to unitary equivalence) to a Haag regime (genuinely inequivalent classes). It is the coupling constant of the SvN $\rightarrow$ Haag transition restricted to data-encoding classes.
- *Q/I diagnostics as field probes.* The negative-volume fraction of the marginal Wigner function (Q) and mode-wise mutual information (I) are properties of the encoded field, not of any data point. They detect, respectively, local non-Gaussianity and nonlocal entanglement: two orthogonal axes of departure from the commutative class.
- *Sandwich* $R_Y \cdot ZZ \cdot R_Y$ *as field interaction.* The sandwich circuit is justified because $\hat{Q}$ and $\hat{P}$ carry field-theoretic meaning (intensity and information flux of bosonic modes). It is not an ansatz; it is the minimal interacting choice within the class of Gaussian-plus-cubic field encodings.

6 Pedagogical consequence

The seminar narrative for a statistical audience falls out automatically.

1. *Act I.* Your data is not a manifold. Geometry is constructed by the choice of encoding map. Probability loading is one particular and trivializing choice.
2. *Act II.* Noncommutativity buys hypothesis-class expansion. Departing from the commutative class is what differentiates classical statistics from quantum data analysis.

3. *Act III.* The Q/I diagnostics measure how far one has departed. They are operational quantities computable from the encoded state, not interpretive overlays.

Classical statistics is the $\hbar_{\mathrm{ZC}} \to 0$ limit of one particular, narrow class of maps. The remainder is open territory.

7 Conclusion

The framework is not metaphysical. It is a re-statement of what mathematicians mean by "object" (Yoneda), what differential geometers mean by "geometry" (Cartan, G-structure), what algebraists mean by "space" (Gelfand, Connes), and what topos theorists mean by "world" (Lawvere). What is new is its consistent application to the data-to-Hilbert-space map as the primary object of quantum machine learning. The Equivalence Theorem and the $\hbar_{\mathrm{ZC}}$-mediated SvN–Haag transition are then specific, predictive instances of an old general program.

References

[1] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer (1998).

[2] R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer (1997).

[3] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic*, Springer (1992).

[4] A. Connes, *Noncommutative Geometry*, Academic Press (1994).

[5] F. W. Lawvere and S. H. Schanuel, *Conceptual Mathematics: A First Introduction to Categories*, 2nd ed., Cambridge University Press (2009).

[6] A. Kock, *Synthetic Differential Geometry*, 2nd ed., Cambridge University Press (2006).

[7] The Univalent Foundations Program, *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study (2013).

[8] J. Earman, *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time*, MIT Press (1989).

[9] C. Rovelli, *Quantum Gravity*, Cambridge University Press (2004).

[10] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955).

[11] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Springer (1992).

[12] S. Zając, "When Is Quantum Data Analysis Actually Quantum? The Inverse Born Rule Fallacy," arXiv:2602.21350 (manuscript v3, 2026).

[13] S. Zając et al., "Quantum Information Field Theory: Diagnostics, Sandwich Circuits, and Active Spectral Recovery," manuscript (2026).